

Tail Behaviour of sums of random components

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Abstract

A class of stochastic processes strongly related to random sums plays an important role in network and in finance. In this paper we study this kind of stochastic process discuss an overtime unchanged parameter and reveal its asymptotic behavior.

1. Introduction

A class of discrete stochastic processes $\{X_n\}$ of the form

$$X_{n+1} = \sum_{i=0}^{X_n} \xi_i^{(n+1)} \quad (1.1)$$

plays an important role in modeling of network [10] and in modeling of finance[8]. In this paper we study this kind of stochastic process, introduce an overtime unchanged parameter to characterize it and reveal its asymptotic behavior. As we see in (1.1), the interesting thing is that here X_n is the counting process for its successor X_{n+1} for each n .

Our main motivation comes from insurance. We recall the classical model of insurance risk [8]. Let u stand for initial capital, c for loaded premium rate and $N(t)$ stands for the number of claims until time t . The total claim amount S_t consists of a random sum of independent and identically distributed claims X_i

$$U(t) = u + ct - S_t, S_t = \sum_{i=1}^{N(t)} X_i, t \geq 0$$

. It is common to simplify this model further by assuming $N(t)$ is a homogeneous Poisson process independent of X_i

As a motivation, we recall a model of network traffic proposed in [10]. Let us consider a chain of routers through which we send two packets. Here we study inter delay time τ_n , time interval between observed packets. We fix a basic time interval, assuming to be 1 which corresponds to a single service time on each

router. If the inter delay time is k , following our assumption, there are $k - 1$ packets between the two observed packets. The inter delay time can be changed in passing the routers influenced by lateral traffic. Here let random variables $\xi_i^{(n)}$ represent the lateral traffic and it can be interpreted as “the sum of the packets entering this chain” while the i -th packet is being served. After the two observed packets pass $n + 1$ -th router, the inter delay time is of the form

$$\tau_{n+1} = \sum_{i=1}^{\tau_n} \xi_i^{(n)}$$

Consider, if no packets enters and leaves the chain on the n -th router, all ξ_i^n take value 1 then $\tau_{n+1} = \tau_n$. If only one packet leave the chain, then $\tau_{n+1} = \tau_n - 1$. So if we set $\xi_i^n \geq 0$, then both “entering” and “leaving” cases are included.

A major goal is to show the limit distribution. Since it is not possible, in general, to find the explicit form of the limit distribution, we will one of most interesting features is the behavoir of the tails.

In this paper we study a class of stochastic processes related to random sum. We will see that our work describing the behavoir of inter delay time, is a generalization of that of U. Sorger and Z. Suchanecki[10] which has heavy tails.

First, we introduce a parameter, which is associated with tail property of a distribution, to characterize this stochastic process. We prove that this parameter is convolution invariant, unchanged over time and it depends only on components random variables. As an extra Bonus we found that for convoluted distribution F the term $\bar{F}(x)e^{xs}$ can never converge to a positive constant, it can only be divergent or converge to 0.

Second, we study the asymptotic property of such stochastic process $\{X_n\}$ and present a way to calculate its limiting distribution.

As mean and variance depend on n , $\{X_n\}$ is not a stationary stochastic process. However, there is a parameter which can be regarded as a numerical characteristic of this stochastic process. Such a parameter will be handled in section 3. We will see that such a parameter can deal with not only discrete case, but also continuous case.

2. Basic properties of this class of stochastic process

First we give a precise definition of this stochastic process $\{X_n\}$. We consider a class of stochastic process X_n equipped with a given stochastic process $\xi_i^{(n)}$ of the form

$$X_{n+1} = \sum_{i=0}^{X_n} \xi_i^{(n+1)} \quad (2.1)$$

such that

1. X_0 is a positive integer constant
2. X_n takes values in set of positive integers
3. $\xi_i^{(n)}$ is a given independent and identically distributed stochastic process

4. X_n and $\xi_i^{(k)}$ are independent for all n, k, i .

(2.1) implies a compound distribution [2], so we have so following properties

Proposition 1. *For this stochastic process $\{X_n\}$ we have*

1. $E[X_{n+1}] = E[X_n] E[\xi]$
2. $\sigma^2[X_{n+1}] = E[X_n] \sigma^2[\xi] + \sigma^2[X_n] E[\xi]^2$
3. $F_{n+1}(y) = \sum_{k=1}^{+\infty} P[X_n = k] F_\xi^{*k}(y)$

where F_n denotes the distribution of X_n and F_ξ the distribution of ξ .

Proof. See [2]. □

Both parameters, mean and variance, depend on n , so $\{X_n\}$ is not stationary. However, there is a parameter unchanged over time, which will be studied in the next section.

3. A convolution invariant Parameter

As we mention in section2 such stochastic process $\{X_n\}$ is not stationary. However, we can characterize this stochastic process with a parameter. In this section we study such a parameter. We define a parameter by

$$C(F) := \sup \left\{ t \in \mathbb{R}_{\geq 0} : \lim_{x \rightarrow +\infty} \overline{F}(x) e^{xt} = 0 \right\} \quad (3.1)$$

where F denotes a given distribution of a random variable and tail $\overline{F} = 1 - F$. First, we prove this parameter is convolution invariant and second, we prove this parameter characterize this stochastic process.

Theorem 2. *The parameter C is the limit of hazard rate function, i.e.*

$$C(F) = \lim_{x \rightarrow \infty} \hat{m}(x)$$

It is not difficult to see [9]

$$\overline{F^{*2}}(x) = \overline{F}(x) + \int_0^x \overline{F}(x-y) dF(y) \quad (3.2)$$

What's more, we have a more general form

$$\overline{F^{*(k+1)}}(x) = \overline{F^{(k)}}(x) + \int_0^x \overline{F^{(k-1)}}(x-y) dF(y), \forall k \in \mathbb{N} \quad (3.3)$$

Before we prove that the parameter C in (3.1) depending on a distribution F is convolution invariant, we need the following lemma.

Lemma 3. If $\lim_{x \rightarrow +\infty} \overline{F}(x)e^{xs} = 0$, then there exists positive s^* such that

$$\overline{F}(x) \leq e^{-xs^*}$$

as long as $x > X$ for some X .

Proof. From the definition of convergence we have $\forall \epsilon > 0, \exists X > 0, \forall x > X$ such that

$$0 < \overline{F}(x)e^{xs} < \epsilon$$

We choose $\epsilon < 1$ and s^* fixed by

$$s^* = s + \frac{\ln \frac{1}{\epsilon}}{X}$$

then

$$\overline{F}(x) \leq \epsilon e^{-xs} \leq \epsilon^{\frac{x}{X}} e^{-xs} = e^{-x(s + \frac{\ln(1/\epsilon)}{X})} \leq e^{-xs^*}$$

The lemma is proved. \square

Proposition 4. If $\overline{F}(x)e^{xs}$ converges to 0 for some $s > 0$, then $\overline{F^{*2}}(x)e^{xs}$ converges to 0.

Proof. First we consider discrete case. We use the same s^* and X in lemma 3 and let $x > 2X$. Then we calculate (3.2) and obtain

$$\begin{aligned} \overline{F^{*2}}(x) &= \overline{F}(x) + \sum_{y=0}^{x-1} \overline{F}(x-y) [\overline{F}(y) - \overline{F}(y+1)] \\ &= \overline{F}(x) + \sum_{y=0}^{x-1} \overline{F}(x-y) \overline{F}(y) \left[1 - \frac{\overline{F}(y+1)}{\overline{F}(y)} \right] \\ &\leq \overline{F}(x) + \sum_{y=0}^X \overline{F}(x-y) \overline{F}(y) + \sum_{y=X+1}^{x-X} \overline{F}(x-y) \overline{F}(y) \\ &\quad + \sum_{y=x-X}^{x-1} \overline{F}(x-y) \overline{F}(y) \\ &\leq \overline{F}(x) + 2 \sum_{y=0}^X e^{-s^*(x-y)} + (x-2X) e^{-s^*x} \\ &= \overline{F}(x) + e^{-s^*x} \left(\frac{2(e^{s^*X} - 1)}{1 - e^{-s^*}} + x - 2X \right) \end{aligned}$$

Finally

$$0 \leq \overline{F^{*2}}(x)e^{xs} \leq \overline{F}(x)e^{xs} + \frac{1}{e^{(s^*-s)x}} \left(\frac{2(e^{s^*X} - 1)}{1 - e^{-s^*}} + x - 2X \right)$$

Due to L'Hôpital's rule the last term converges to 0 as $x \rightarrow +\infty$. Therefore $\overline{F^{*2}}(x)e^{sx}$ converges to 0.

Second we consider continuous case and let $\lfloor x \rfloor$ denote the integer part of x , then

$$\begin{aligned}\overline{F^{*2}}(x) &= \overline{F}(x) + \int_0^x \overline{F}(x-y) dF(y) \\ &\leq \overline{F}(x) + \sum_{y=0}^{\lfloor x \rfloor - 1} \overline{F}(x-y) [\overline{F}(y) - \overline{F}(y+1)] \\ &\leq \overline{F}(x) + e^{-s^* \lfloor x \rfloor} \left(\frac{2(e^{s^* X} - 1) - e^{-s^* \lfloor x \rfloor}}{1 - e^{-s^*}} + \lfloor x \rfloor - 2X \right)\end{aligned}$$

Analog, $\overline{F^{*2}}(x)e^{xs}$ converges to 0. So far both discrete case and continuous case are considered. This proposition is proved. \square

In the following we prove a more general proposition.

Proposition 5. *If $\overline{F}(x)e^{xs}$ converges to 0 for some $s > 0$, then $\overline{F^{*k}}(x)e^{xs}$ converges to 0 for any natural number k .*

Proof. We prove it by mathematical induction. Assume $\overline{F^{*k}}(x)e^{xs}$ converges to 0, what we need to show, due to (3.3), is

$$\lim_{x \rightarrow +\infty} e^{xs} \left[\sum_{y=0}^{x-1} \overline{F^{*k}}(x-y) (\overline{F}(y) - \overline{F}(y+1)) \right] = 0$$

We know from lemma 3, there exist positive number s_1 and X_1 such that for all

$$\overline{F}(x) < e^{xs_1}$$

as long as $x > X_1$.

Analog there exist positive number s_2 and X_2 such that for all

$$\overline{F}(x) < e^{xs_2}$$

as long as $x > X_2$.

Let $X = \max\{X_1, X_2\}$ and $s^* = \min\{s_1, s_2\}$, then

$$\begin{aligned}0 &\leq e^{xs} \left[\sum_{y=0}^{x-1} \overline{F^{*k}}(x-y) (\overline{F}(y) - \overline{F}(y+1)) \right] \\ &\leq \frac{x - 2X + 2 \left(\frac{e^X - 1}{1 - e^{s^*}} \right)}{e^{(s^* - s)x}}\end{aligned}$$

Again the last term converges to 0 as $x \rightarrow +\infty$, hence $\overline{F^{*(k+1)}}(x)e^{xs}$ converges to 0. The proposition is proved. \square

Second we will prove a somehow surprising result that for a convoluted distribution the term $\overline{F^{*k}}(x)e^{xs}$ can never converge to a positive constant, which can be formulated in this proposition.

Proposition 6. *If $\overline{F}(x)e^{xs}$ tends to a positive constant c for some s , then $\overline{F^{*2}}(x)e^{xs}$ tends to $+\infty$ for all $k \geq 2$.*

Proof. Let $\overline{F}(x)e^{xs}$ converges to a positive constant C . We see two inequalities. For $\forall \epsilon > 0, \exists X > 0, \forall x > X$ the inequality

$$C - \epsilon < e^{sx}\overline{F}(x) < C + \epsilon$$

the inequality

$$\overline{F}(x)e^{xs} \geq \min \{ \overline{F}(0), \overline{F}(1)e^{1s}, \dots, \overline{F}(X)e^{Xs}, C - \epsilon \}$$

Let

$$m := \min \{ \overline{F}(0), \overline{F}(1)e^{1s}, \dots, \overline{F}(X)e^{Xs}, C - \epsilon \}$$

What we need to prove is that

$$\lim_{x \rightarrow \infty} \frac{\sum_{y=0}^{x-1} \overline{F}(x-y) [F(y) - F(y+1)]}{\overline{F}(x)} = +\infty$$

Then we estimate

$$\begin{aligned} \frac{\sum_{y=0}^{x-1} \overline{F}(x-y) [F(y) - F(y+1)]}{\overline{F}(x)} &\geq \frac{\sum_{y=X+1}^{x-1} \overline{F}(x-y) [F(y) - F(y+1)]}{\overline{F}(x)} \\ &\geq \frac{\sum_{y=0}^{x-1} \overline{F}(x-y)e^{s(x-y)} \left[F(y)d^{sy} - \frac{F(y+1)e^{s(y+1)}}{e^s} \right]}{\overline{F}(x)e^{xs}} \\ &\geq \sum_{y=X+1}^{x-1} \frac{m \left[\left(1 - \frac{1}{e^s}\right) C - \epsilon \left(1 + \frac{1}{e^s}\right) \right]}{C + \epsilon} \end{aligned}$$

The last term tends to infinity as $x \rightarrow +\infty$. The proposition is proved. \square

Combining propositions (4) (5) and (6), we can obtain the following theorem

Theorem 7. *Parameter $C(F)$ defined through*

$$C(F) := \sup \left\{ t \in \mathbb{R}_{\geq 0} : \lim_{x \rightarrow +\infty} \overline{F}(x)e^{xt} = 0 \right\}$$

is invariant under convolution, i.e.

$$C(F) = C(F^{*k})$$

for all natural number k .

Example. We consider exponential distribution

$$F(x) = 1 - e^{-\lambda x}$$

and its convolution

$$F^{*2}(x) = 1 - e^{-\lambda x} (1 + \lambda x)$$

Obviously

$$C(F) = C(F^{*2}) = \lambda$$

Theorem 8. *The parameter of X_n in (2.1) depends only on ξ , what's more, for all $n > 0$*

$$C(F_n) = C(F_\xi)$$

Proof. From (author?) [6] we have

$$\overline{F}_n(x) = \sum_{k=1}^{+\infty} P[X_{n-1} = k] \overline{F}_\xi^{*k}(x)$$

then

$$\overline{F}_n(x)e^{xs} = \sum_{k=1}^{+\infty} P[X_{n-1} = k] \overline{F}_\xi^{*k}(x)e^{xs}$$

If $\overline{F}_\xi^{*k}(x)e^{xs}$ converges to 0 for some $s > 0$, $\overline{F}_n(x)e^{xs}$ converges 0. Hence this parameter C depends only on ξ while is independent of n . \square

4. Asymptotic properties

In this section we study the asymptotic properties of $\{X_n\}$ and denote the limiting probability function by f^* .

We recall

$$F_{n+1}(y) = \sum_{k=1}^{+\infty} P[X_n = k] F_\xi^{*k}(y) \quad (4.1)$$

and further

$$F_{n+1}(y-1) = \sum_{k=1}^{+\infty} P[X_n = k] F_\xi^{*k}(y-1) \quad (4.2)$$

Letting (4.1)-(4.2), then

$$f_{n+1}(y) = \sum_{k=1}^{+\infty} f_n(k) f_\xi^{*k}(y)$$

where f_n denotes the probability function of X_n and f_ξ the probability function of ξ .

We consider $\left(f_\xi^{*k}(y)\right)_{y,k}$ as a Markov operator, which maps a probability function to another, and denote it by M_ξ , then the above expression can be rewritten in form of

$$f_{n+1}(y) = (M_\xi f_n)(y)$$

and furthermore

$$f_{n+1}(y) = \left(M_\xi^{n+1} f_0\right)(y)$$

So for the asymptotic property of $\{X_n\}_n$, we have the following proposition.

Proposition 9. f_n converges to a function f^* satisfying

$$\sum_{k \neq j} f^*(k) f_\xi^{*k}(j) + f^*(j) (1 - f_\xi^{*j}(j)) = 0 \quad (4.3)$$

Proof. Operator M_ξ is of the form

$$M_\xi = \sum_{i=1} \lambda_i E_{\lambda_i}$$

and further

$$M_\xi^n = \sum_{\lambda_i \in \sigma(M_\xi)} \lambda_i^n E_{\lambda_i} \quad (4.4)$$

where λ_i denote the eigenvalues of M_ξ and E_i denote the projections projecting the entire space into eigenspaces corresponding to λ_i .

We apply Gelfand theorem [?] to here

$$\sigma(M_\xi) = \lim_{n \rightarrow \infty} \|M_\xi^n\|^{\frac{1}{n}} = 1$$

In other words, $|\lambda_i|$ is bound by 1 for all eigenvalues.

Let n tend to infinity and we see that only eigenvalue $\lambda_i = 1$ contributes in the right hand side of (4.4), therefore this limiting probability function f^* is a fix-point of M_ξ , i.e.

$$M_\xi f^* = f^*$$

After simplification we obtain (4.3) and the proposition is proved. \square

5. Conclusion

In this paper we study a class of stochastic processes, which is widely used in modeling of Network and Finance and obtain its two asymptotic properties.

Our main aim

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